Overview

1) The lecture will focus on "topological quatum field theories" (TQFTs) in 1+1d (chiral conformal field theory, conformal blocks) and 2+1d (CS-theory, Jones polynomial).

2) A TQFT is a quantum field theory which computes "topological invariants"

→ correlation/partition functions do not depend on metric of spacetime

(In a chiral CFT they do depend on the complex structure, so the above notion has to be relaxed somewhat.)

→ TQFTs are applied to curved spacetimes, such as Riemann surfaces Σ and 3-manifolds M.

3) Applications : Non-abelian anyons

- quantum statistics in 3+1D :
  interchanging two particles twice gives

$\psi(\vec{r}_1, \vec{r}_2)$ is invariant

(a) (b)

$\Rightarrow$ under 1x interchange we have:

$$\psi(\vec{r}_1, \vec{r}_2) \rightarrow \pm \psi(\vec{r}_1, \vec{r}_2)$$

$+$: bosons

$-$: fermions

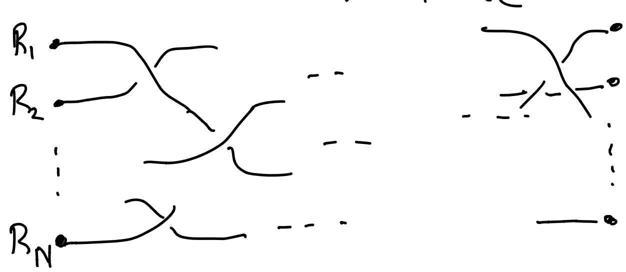
- quantum statistics in 2+1D :

$$\psi(\vec{r}_1, \vec{r}_2) \rightarrow e^{i\theta} \psi(\vec{r}_1, \vec{r}_2) \qquad \big((a) \not\Leftrightarrow (b) \big)$$

$\rightarrow$ "anyons"

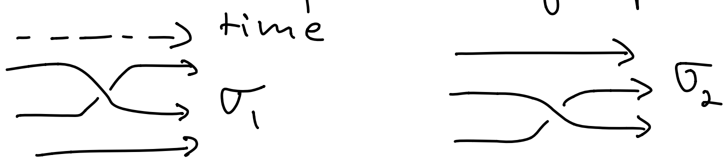
general case of N particles:

------> time



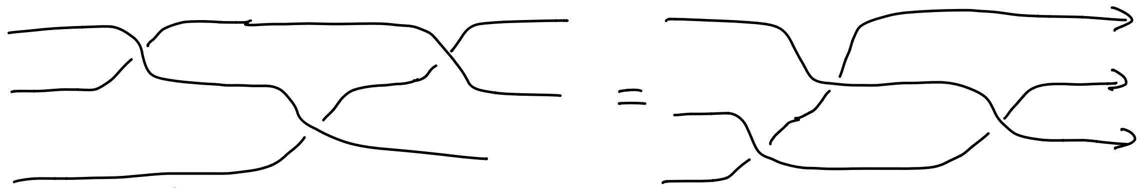
$\rightarrow$ "braid group" B_N

generators of braid group:



for 3 particles. In general $\sigma_1, \dots, \sigma_{N-1}$

braid relation:



Important relations in a TQFT

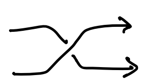
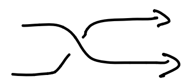
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

also: $\sigma_i \sigma_j = \sigma_j \cdot \sigma_i$ for $|i-j| \geq 2$

$\sigma_i^2 \neq 1 \Rightarrow$ infinitely many elements

- abelian representations of B_N:

$$\psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) \longrightarrow e^{im\theta} \psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)$$

where $m = \#$  $- \#$ 

for identical particles. Non-identical case:

θ_{ab}, $a, b = 1, \dots, n_s$

$\uparrow$ number of particle species

- non-abelian repr. of B_N:

degenerate states

$\longrightarrow$ g states with particles at $R_1, \dots, R_n$

$$\psi_\alpha, \ \alpha = 1, 2, \dots, g$$

Then σ_i act as $g \times g$ unitary matrix $\rho(\sigma_i)$

$$\psi_\alpha \to [\rho(\sigma_1)]_{\alpha\beta}\,\psi_\beta$$

in particular: $\rho(\sigma_1)\rho(\sigma_2) \neq \rho(\sigma_2)\rho(\sigma_1)$

$\to$ "non-abelian braiding statistics"

<u>Example:</u>

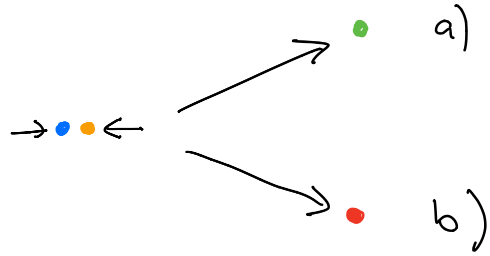
Consider a model with 3 anyon types:

$1, \sigma, \psi$

with "fusion rules":

$$\sigma \times \sigma = 1 + \psi, \quad \sigma \times \psi = \sigma, \quad \psi \times \psi = 1,$$

$$1 \times x = x \quad \text{for } x = 1, \sigma, \psi$$



fusion channels

Note the similarity to tensor products of SU(2) representations:

$$\underset{\sigma}{\tfrac{1}{2}} \times \underset{\sigma}{\tfrac{1}{2}} = \underset{1}{0} + \underset{\psi}{1}, \quad \underset{\sigma}{\tfrac{1}{2}} \times \underset{\psi}{1} = \underset{\sigma}{\tfrac{1}{2}}, \quad \underset{\psi}{1} \times \underset{\psi}{1} = \underset{1}{0}$$

with important <u>constraint</u>: maximum spin $= 1$

$\to$ Will see how these fusion rules arise in a TQFT later on.

Hilbert space of 4 σs:

group according to (1,2) and (3,4)

$$(1,2) \to 1, \ \psi \qquad (3,4) \to \ , \ \psi$$

constraint: global topological charge $= 1$

$\Rightarrow$ σ_1 and σ_2 fuse to 1 (σ_3 and σ_4 too)

or σ_1 and σ_2 fuse to ψ (σ_3 and σ_4 too)

$\Rightarrow$ two-dim Hilbert space generated by Ψ_1 and Ψ_ψ

In general: for $2n$ quasi-particles Hilbert space is 2^{n-1} dimensional.

action of braid group generators: spinor representation of $SO(2n)$!

braiding particles i and j $\to$ $\frac{\pi}{2}$ rotation in the i-j plane of $\mathbb{R}^{2n}$

Realization in TQFT:

We will see in the course of these lectures that wave-functions can equivalently be described in terms of correlation functions of TQFTs:

$$\psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = \langle O_1(\vec{r}_1) \cdots O_N(\vec{r}_N) \rangle_{TQFT} \quad \text{“vector”}$$

4) Abstract definition of 2+1d TQFTs

A TQFT in 2+1 dimensions (more general D+1 dimensions) is a functor Z satisfying the following conditions

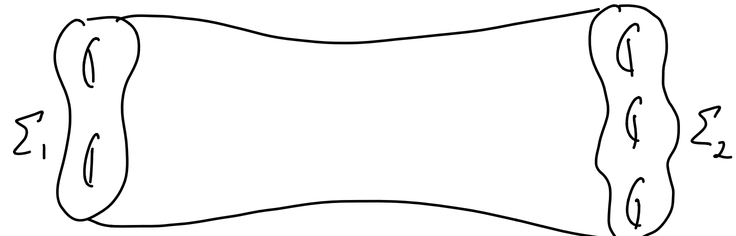
1. For each compact oriented Riemann surface Σ without boundary
   $\longrightarrow$ complex vector space Z_Σ.
2. A compact oriented 3-dimensional smooth manifold Y with $\partial Y = \Sigma$ determines a vector $Z(Y) \in Z_\Sigma$.

Furthermore, Z satisfies the following axioms:

(A1) Let $-\Sigma$ be Σ with reverse orientation
$\longrightarrow$ $Z_{-\Sigma} = Z_\Sigma^*$ (dual vector space)

(A2) $Z_{\Sigma_1 \cup \Sigma_2} = Z_{\Sigma_1} \otimes Z_{\Sigma_2}$

A 3-manifold Y with $\partial Y = (-\Sigma_1) \cup \Sigma_2$
$\longrightarrow$ linear map $Z(Y) \in \mathrm{Hom}(Z_{\Sigma_1}, Z_{\Sigma_2})$



"cobordism"

$$Z_{\Sigma_1} \xmapsto{Z(Y)} Z_{\Sigma_2}$$

(A3) $\partial Y_1 = (-\Sigma_1) \cup (\Sigma_2)$ and $\partial Y_2 = (-\Sigma_2) \cup \Sigma_3$

$\rightarrow Z(Y_1 \cup Y_2) = Z(Y_2) \circ Z(Y_1)$

(A4) For an empty set $\emptyset$ we have $Z(\emptyset) = \mathbb{C}$

(A5) Let I denote the closed unit interval. Then, $Z(\Sigma \times I)$ is the identity map as a linear transformation of Z_Σ.

<u>Lecture Content</u>

1) Conformal Field Theory & Topology
- Loop groups and affine Lie algebras
- Representations of affine Lie algebras
- Wess-Zumino Witten model
- The space of conformal blocks
- KZ equation
- Vertex operators and OPE

2) Chern-Simons Theory

- KZ equations and representations of braid groups
- Conformal field theory and the Jones polynomial
- Witten's invariants for 3-manifolds
- Projective representations of mapping class groups
- Chern-Simons theory and connections on surfaces

3) Applications of CS-theory: Non-Abelian anyons

- Non-Abelian braiding statistics
- Emergent Anyons
- Review of Quantum Hall Physics
- Quantum Hall wave-functions from conformal field theory